

Lesson 1.1 · Worked solutions

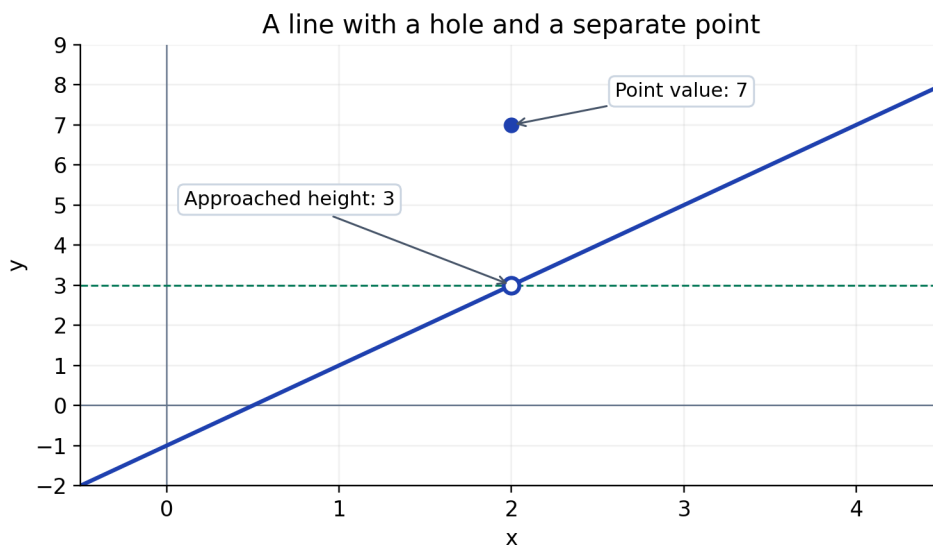
Show the method, check hypotheses, and justify your conclusion.

1. For $f(x) = \frac{x^2 - 9}{x - 3}$, $x \neq 3$, make a table at $x = 2.9, 2.99, 3.01, 3.1$. Estimate $\lim_{x \rightarrow 3} f(x)$.

Solution. The outputs are 5.9, 5.99, 6.01, 6.1. Both sides suggest 6. Factoring gives $f(x) = x + 3$ for $x \neq 3$, so $\lim_{x \rightarrow 3} f(x) = 6$.

2. Use the graph of f below. It follows $y = 2x - 1$, with an open circle at $(2, 3)$ and a filled dot at $(2, 7)$. Find $\lim_{x \rightarrow 2^-} f(x)$, $\lim_{x \rightarrow 2^+} f(x)$, $\lim_{x \rightarrow 2} f(x)$, and $f(2)$.

Solution. Tracing the line toward $x = 2$ from either side gives $\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = 3$. Therefore $\lim_{x \rightarrow 2} f(x) = 3$. The filled dot gives $f(2) = 7$; its height does not change the limit.



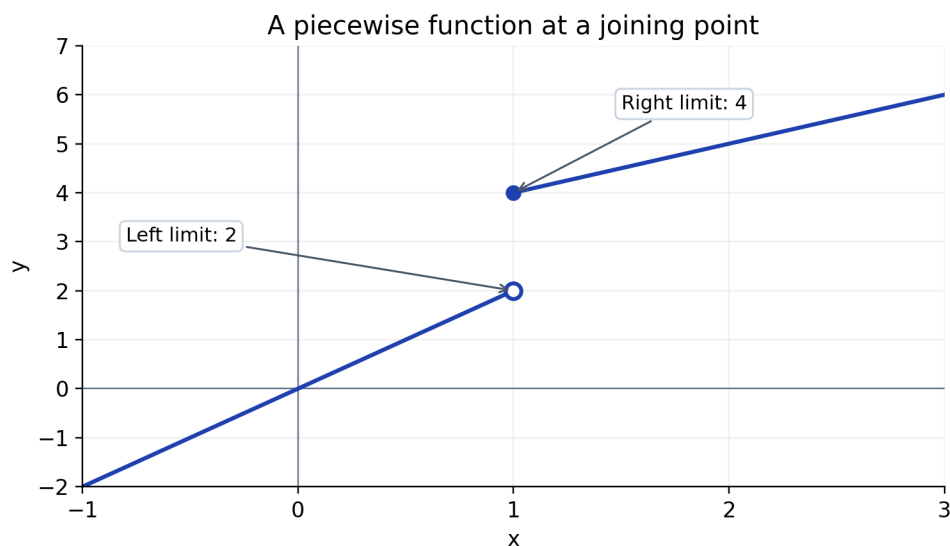
Both sides approach height 3; the filled point gives $f(2) = 7$.

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Show the method, check hypotheses, and justify your conclusion.

3. Let $p(x) = 2x$ for $x < 1$ and $p(x) = x + 3$ for $x \geq 1$. Use the graph and the formulas to find both one-sided limits at 1. Does $\lim_{x \rightarrow 1} p(x)$ exist?

Solution. The left branch approaches 2, while the right branch approaches 4. Thus $\lim_{x \rightarrow 1^-} p(x) = 2$ and $\lim_{x \rightarrow 1^+} p(x) = 4$. Since these differ, the two-sided limit does not exist, even though $p(1) = 4$.



The left and right approached heights differ, so there is no two-sided limit.

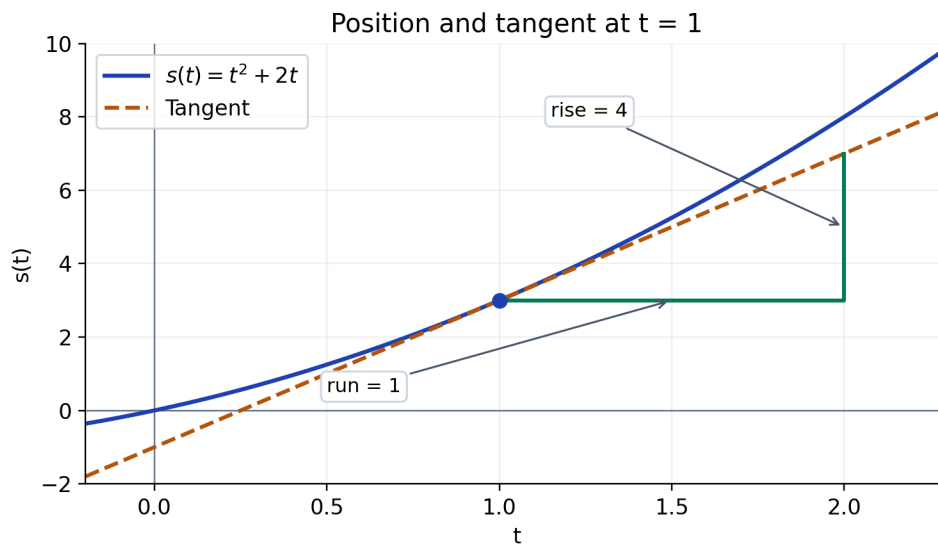
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Show the method, check hypotheses, and justify your conclusion.

4. For $s(t) = t^2 + 2t$, the average velocity on $[1, 1 + h]$ is $\frac{s(1+h) - s(1)}{h} = 4 + h$ for $h \neq 0$. Evaluate it for $h = 0.1, 0.01, -0.01$. Find the instantaneous velocity $s'(1)$.

Solution. The averages are 4.1, 4.01, 3.99. The derivative is the limit of these average rates:

$s'(1) = \lim_{h \rightarrow 0} \frac{s(1+h) - s(1)}{h} = \lim_{h \rightarrow 0} (4 + h) = 4$. The tangent line at $(1, 3)$ has slope 4, in distance units per time unit.



The tangent has rise 4 over run 1, matching the derivative s' of $1 = 4$.

Lesson 1.2 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

5. Evaluate $\lim_{x \rightarrow -1} (2x^2 - 3x + 4)$. Explain why substitution works.

Solution. A polynomial is continuous at -1 , so direct substitution gives $2(-1)^2 - 3(-1) + 4 = 9$. The limit is 9.

6. Evaluate $\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$. Show the factorization and restriction.

Solution. Factor $x^2 - 16 = (x - 4)(x + 4)$. For $x \neq 4$, the quotient equals $x + 4$. Thus

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4} = \lim_{x \rightarrow 4} (x + 4) = 8.$$

7. Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{16+x} - 4}{x}$. Show the conjugate step.

Solution. Multiply by $\frac{\sqrt{16+x} + 4}{\sqrt{16+x} + 4}$. The numerator becomes x . For $x \neq 0$, cancellation gives

$$\frac{1}{\sqrt{16+x} + 4}. \text{ Substitution now gives } \frac{1}{4+4} = \frac{1}{8}.$$

8. Evaluate $\lim_{x \rightarrow 2} \frac{x^2 - 5x + 6}{x^2 - 4}$. Explain why the function need not be defined at 2.

Solution. Factor to get $\frac{(x-2)(x-3)}{(x-2)(x+2)}$. For nearby inputs with $x \neq 2$, this equals $\frac{x-3}{x+2}$. Its limit is

$$\frac{2-3}{2+2} = -\frac{1}{4}. \text{ A limit depends on nearby values rather than the point value.}$$

Lesson 1.3 · Worked solutions

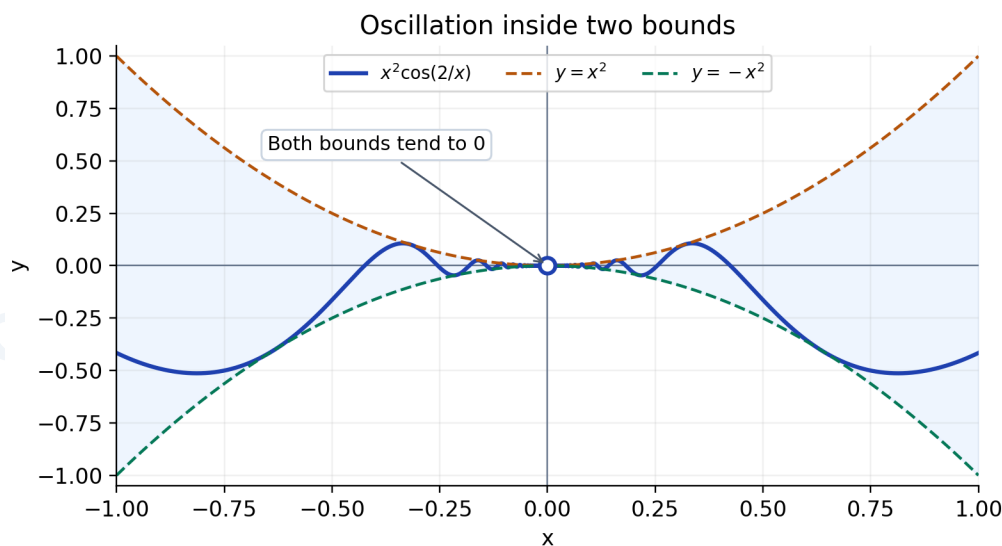
Show the method, check hypotheses, and justify your conclusion.

9. Suppose $4 - x^2 \leq f(x) \leq 4 + 2x^2$ near 0. Find $\lim_{x \rightarrow 0} f(x)$ and name the theorem.

Solution. Both bounding functions approach 4: $\lim_{x \rightarrow 0} (4 - x^2) = \lim_{x \rightarrow 0} (4 + 2x^2) = 4$. Since the inequalities hold near 0, the Squeeze Theorem gives $\lim_{x \rightarrow 0} f(x) = 4$.

10. Evaluate $\lim_{x \rightarrow 0} x^2 \cos\left(\frac{2}{x}\right)$. Justify your answer with bounds.

Solution. Since $-1 \leq \cos(2/x) \leq 1$ and $x^2 \geq 0$, we have $-x^2 \leq x^2 \cos(2/x) \leq x^2$. Both bounds approach 0. The Squeeze Theorem gives $\lim_{x \rightarrow 0} x^2 \cos(2/x) = 0$, even though the cosine factor oscillates.



Both envelopes approach zero, forcing the middle function toward zero.

11. Evaluate $\lim_{x \rightarrow 0} \frac{\sin(6x)}{4x}$ in radians. Rewrite it using the standard trigonometric limit.

Solution. Rewrite as $\frac{6}{4} \cdot \frac{\sin(6x)}{6x}$. As $x \rightarrow 0$, the ratio approaches 1. The limit is $\frac{6}{4} = \frac{3}{2}$.

Lesson 1.3 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

12. You know only that $-1 \leq q(x) \leq 1$ near 0. Must $\lim_{x \rightarrow 0} q(x)$ exist? Give a counterexample.

Solution. No: the two bounds have different limits. The function $q(x) = \sin(1/x)$ stays between them but has no limit at 0. There are sequences of inputs tending to 0 for which the outputs stay at 1, and others for which they stay at -1 .

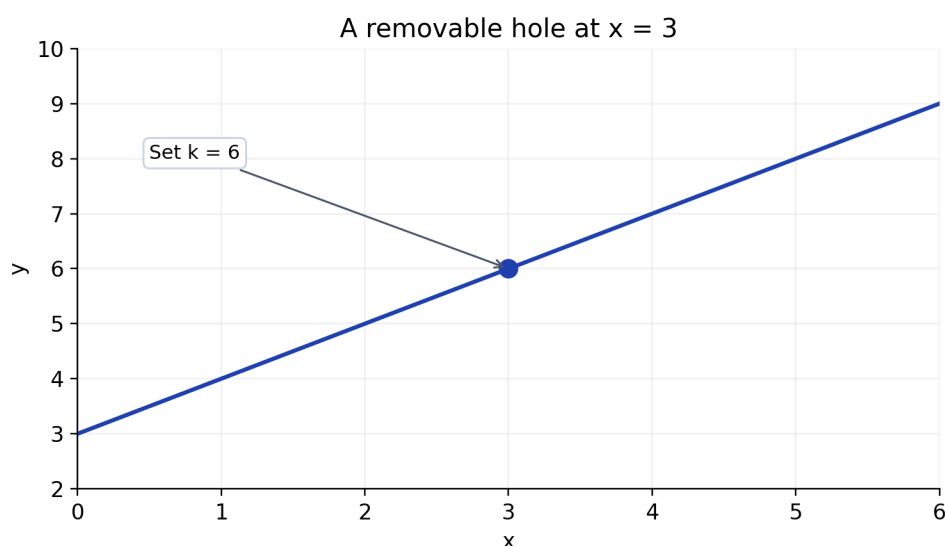
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Lesson 1.4 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

13. Let $f(x) = \frac{x^2 - 9}{x - 3}$ for $x \neq 3$, and $f(3) = k$. Find k for continuity. Classify the break if $k = 0$.

Solution. For $x \neq 3$, cancellation gives $f(x) = x + 3$. Hence $\lim_{x \rightarrow 3} f(x) = 6$. Continuity requires $f(3) = 6$, so choose $k = 6$. If $k = 0$, the finite limit differs from the point value, so the discontinuity is removable.



Filling the hole at height 6 makes the function continuous.

14. Let $f(x) = ax + 2$ for $x < 1$, and $f(x) = x^2 + 4$ for $x \geq 1$. Find a so that f is continuous at 1.

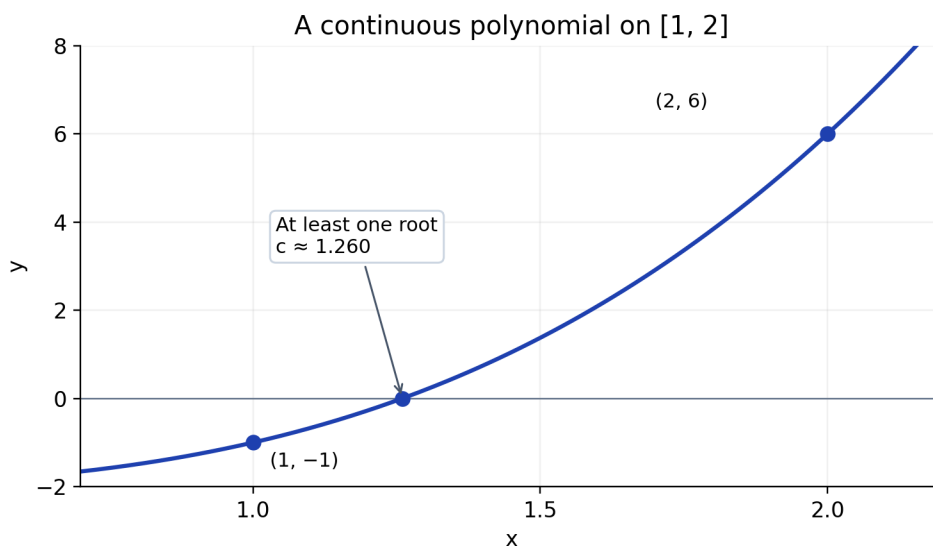
Solution. The left-hand limit is $a + 2$. The right-hand limit and $f(1)$ are both 5. Set $a + 2 = 5$, giving $a = 3$. The two one-sided limits and the point value then agree.

Lesson 1.4 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

15. Use the Intermediate Value Theorem to justify a root of $f(x) = x^3 - 2$ between 1 and 2. Does IVT alone prove uniqueness?

Solution. The polynomial is continuous on $[1, 2]$. Since $f(1) = -1$ and $f(2) = 6$, we have $-1 < 0 < 6$. IVT guarantees at least one $c \in (1, 2)$ with $f(c) = 0$. IVT alone does not establish uniqueness.



The endpoint values bracket zero. IVT guarantees a crossing; the marked root is approximate.

16. Explain why $\sqrt{x+2}$ is continuous on $[-2, 2]$. Then explain why IVT cannot be applied to $\frac{1}{x-1}$ on $[0, 2]$.

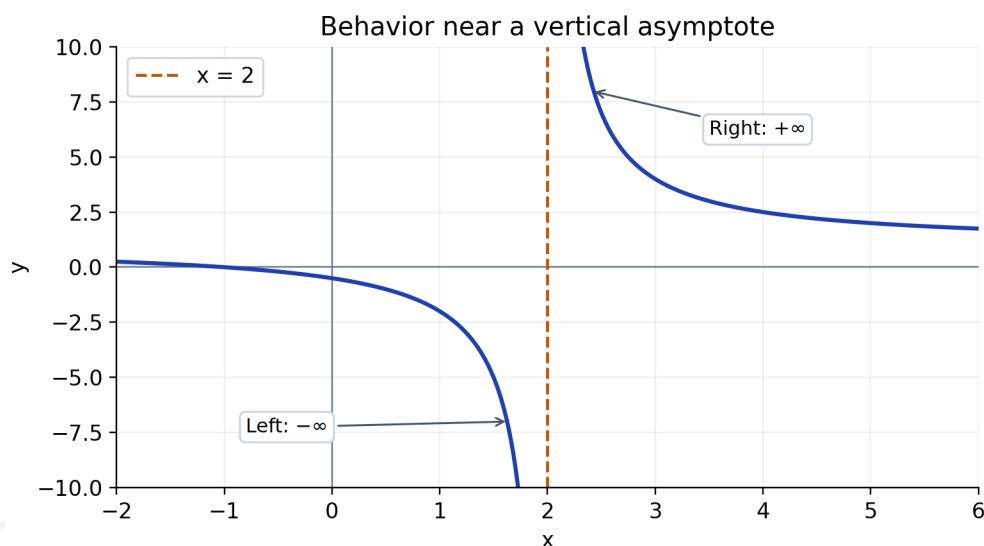
Solution. The square-root function is continuous in the interior, and its endpoint limits agree with its values: $\lim_{x \rightarrow -2^+} \sqrt{x+2} = 0$ and $\lim_{x \rightarrow 2^-} \sqrt{x+2} = 2$. In contrast, $\frac{1}{x-1}$ is undefined at 1, so it is not continuous on $[0, 2]$. The IVT hypothesis fails.

Lesson 1.5 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

17. Find $\lim_{x \rightarrow 2^-} \frac{x+1}{x-2}$ and $\lim_{x \rightarrow 2^+} \frac{x+1}{x-2}$. Does the two-sided limit exist? Identify the vertical asymptote.

Solution. The numerator approaches $3 > 0$. The denominator is negative to the left and positive to the right, so $\lim_{x \rightarrow 2^-} \frac{x+1}{x-2} = -\infty$ and $\lim_{x \rightarrow 2^+} \frac{x+1}{x-2} = +\infty$. The two-sided limit does not exist. The vertical asymptote is $x = 2$.



The left branch tends to negative infinity and the right branch to positive infinity.

18. Evaluate $\lim_{x \rightarrow +\infty} \frac{5x^2 - 3}{2x^2 + 7x}$. Identify the horizontal asymptote.

Solution. Divide by x^2 to obtain $\frac{5 - 3/x^2}{2 + 7/x}$. The reciprocal terms approach 0, so the limit is $\frac{5}{2}$. The horizontal asymptote is $y = \frac{5}{2}$.

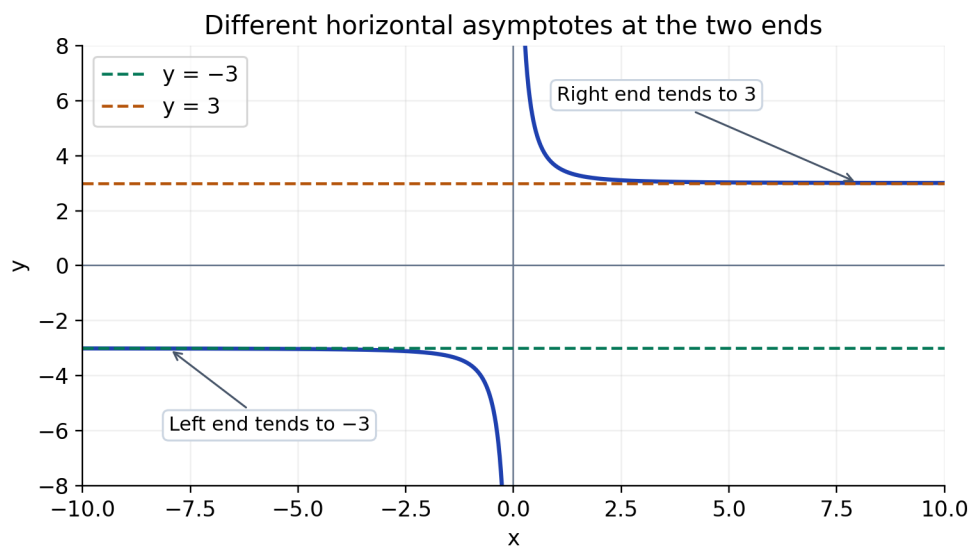
Lesson 1.5 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

19. Evaluate $\lim_{x \rightarrow -\infty} \frac{\sqrt{9x^2 + 4}}{x}$. Explain the sign.

Solution. Rewrite the quotient as $\frac{|x|}{x} \sqrt{9 + \frac{4}{x^2}}$. For negative x , $\frac{|x|}{x} = -1$, while the radical approaches 3.

Therefore the limit is -3 .



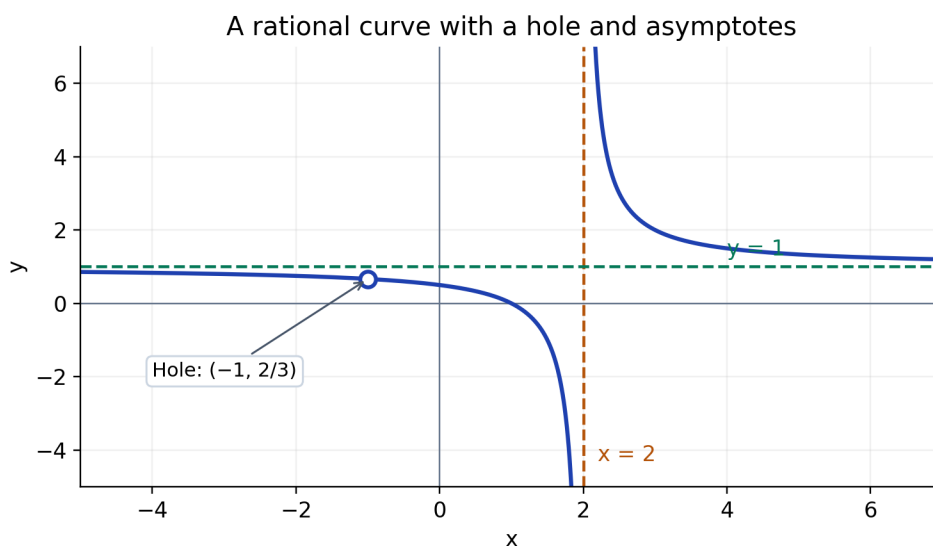
At negative infinity, the curve approaches $y = \text{minus } 3$ because absolute x over x equals minus 1.

Lesson 1.5 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

20. For $f(x) = \frac{x^2 - 1}{x^2 - x - 2}$, classify the excluded inputs and find both end limits.

Solution. Factor to get $\frac{(x-1)(x+1)}{(x-2)(x+1)}$. The excluded inputs are -1 and 2 . At -1 there is a removable hole with limit $\frac{2}{3}$. At 2 a denominator factor remains and the numerator is nonzero, giving a vertical asymptote $x = 2$. Equal leading degrees give $\lim_{x \rightarrow \pm\infty} f(x) = 1$, so $y = 1$ is a horizontal asymptote.



The hole is at (minus 1, two thirds), with asymptotes $x = 2$ and $y = 1$.