

Lesson 3.1 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

1. A sphere has $V = \frac{4}{3}\pi r^3$. If $r = 2$ cm and $\frac{dr}{dt} = 3$ cm/s, find $\frac{dV}{dt}$.

Solution. Differentiate to obtain $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$. At the specified instant the rate is 48π cubic centimeters per second.

2. A 5 m ladder has base distance $x = 3$ m increasing at $\frac{dx}{dt} = 2$ m/s. Find the height rate.

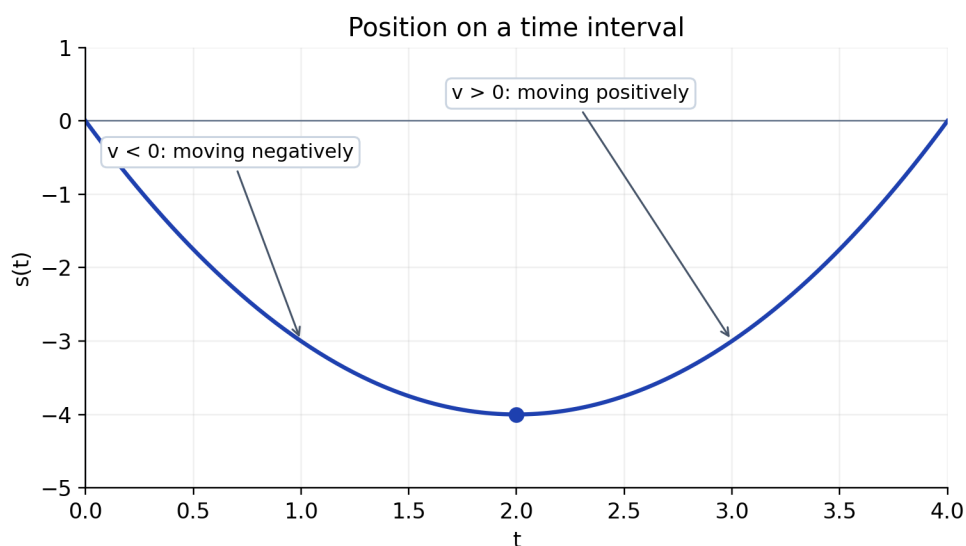
Solution. From $x^2 + y^2 = 25$, the height is $y = 4$. Differentiate: $2x \frac{dx}{dt} + 2y \frac{dy}{dt} = 0$. Thus $\frac{dy}{dt} = -3/2$ m/s: the top is descending.

Lesson 3.1 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

3. For $s(t) = t^2 - 4t$ on $[0, 4]$, describe direction and speed changes.

Solution. Velocity is $2t - 4$ and acceleration is 2 . On $(0, 2)$ the object moves negatively and slows down; on $(2, 4)$ it moves positively and speeds up. It is at rest at $t = 2$.



Negative slopes before $t = 2$ become positive after; acceleration is positive throughout.

Lesson 3.2 · Worked solutions

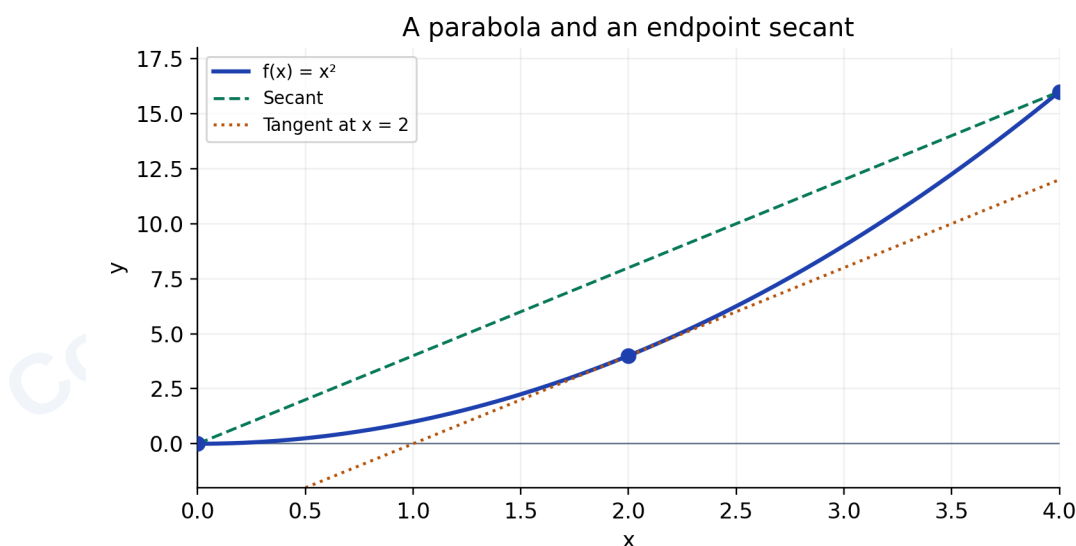
Show the method, check hypotheses, and justify your conclusion.

4. Find absolute extrema of $f(x) = x^3 - 3x$ on $[-2, 2]$.

Solution. The critical numbers are -1 and 1 . Candidate values are $f(-2) = -2$, $f(-1) = 2$, $f(1) = -2$, $f(2) = 2$. The minimum is -2 at -2 and 1 ; the maximum is 2 at -1 and 2 .

5. Find the MVT input for $f(x) = x^2$ on $[0, 4]$ and check its hypotheses.

Solution. The polynomial is continuous on $[0, 4]$ and differentiable on $(0, 4)$. Average slope is 4, so $2c = 4$ gives $c = 2$.



The tangent at $x = 2$ has slope 4, matching the secant.

6. Can MVT be applied to $f(x) = |x|$ on $[-1, 1]$?

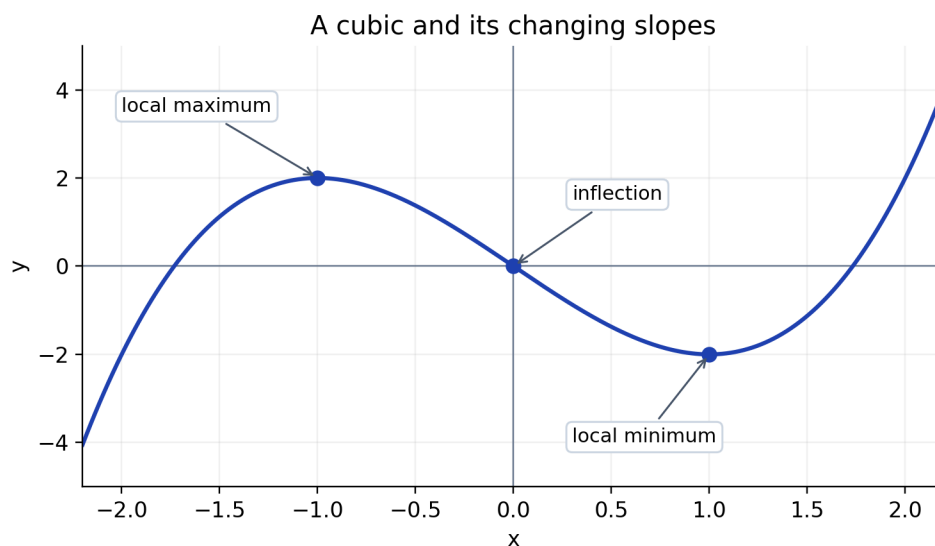
Solution. No. The function is continuous on the closed interval but is not differentiable at 0 , an interior point. Its average slope is zero, yet neither linear branch has zero derivative.

Lesson 3.3 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

7. Analyze direction and concavity for $f(x) = x^3 - 3x$. Locate local extrema and the inflection point.

Solution. The first derivative is $3(x^2 - 1)$: increase on $(-\infty, -1)$ and $(1, \infty)$, decrease on $(-1, 1)$. Local maximum is $(-1, 2)$ and local minimum is $(1, -2)$. The second derivative is $6x$, so the graph is concave down for $x < 0$, up for $x > 0$, with inflection at $(0, 0)$.



Local maximum is (minus 1, 2), local minimum is (1, minus 2), with inflection at the origin.

8. If f' is positive and decreasing on an interval, describe f .

Solution. The function increases because its derivative is positive. Its slopes decrease, so it is concave down where this derivative behavior holds.

Lesson 3.3 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

9. Does $f''(0) = 0$ prove an inflection point? Use $f(x) = x^4$.

Solution. No. Here $f''(x) = 12x^2$ is positive on either side of 0. Concavity does not change, so 0 is not an inflection input.

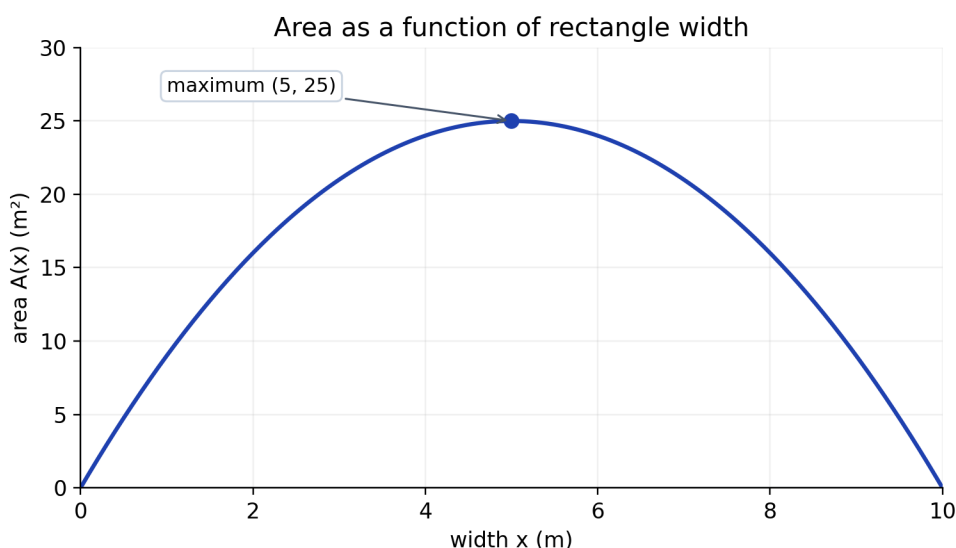
CodeScholarsTutoring

Lesson 3.4 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

10. A rectangle has perimeter 20 m. Find the maximum area and justify it.

Solution. The area is $A(x) = 10x - x^2$ for $0 < x < 10$. Its derivative changes from positive to negative at $x = 5$, so the maximum is 25 square meters for dimensions 5 by 5.



The area is maximal at width 5, giving area 25.

11. Minimize $C(x) = x + 9/x$ on $x > 0$.

Solution. The derivative is $1 - 9/x^2$, negative below 3 and positive above 3. Thus the absolute minimum occurs at $x = 3$ and is $C(3) = 6$.

12. Find the point on $y = x^2$ closest to $(0, 1/2)$.

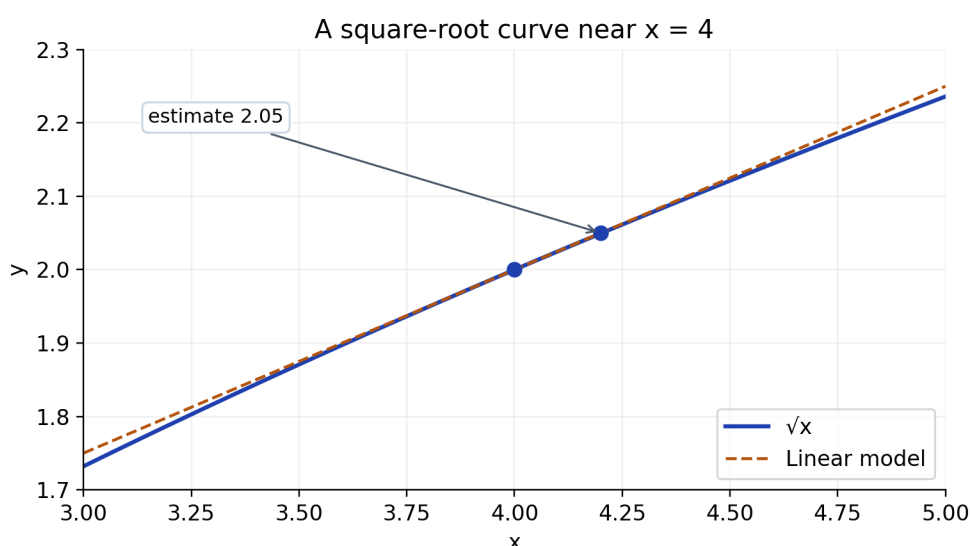
Solution. Minimize squared distance $D(x) = x^2 + (x^2 - 1/2)^2 = x^4 + 1/4$. This is minimized at $x = 0$, so the point is $(0, 0)$ and the distance is $1/2$. Squared distance has the same minimizer as distance.

Lesson 3.5 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

13. Use the tangent at 4 to estimate $\sqrt{4.2}$. Is it above or below the actual value?

Solution. The model is $L(x) = 2 + (x - 4)/4$, giving 2.05. The function is concave down on the interval, so this is an overestimate.



The tangent at $x = 4$ lies above the concave-down curve and predicts 2.05 at $x = 4.2$.

14. Evaluate $\lim_{x \rightarrow 0} \frac{\ln(1+x)}{x}$. State the initial form.

Solution. The form is $0/0$. L'Hôpital gives the derivative quotient $1/(1+x)$, whose limit is 1.

15. Can L'Hôpital's Rule evaluate $\lim_{x \rightarrow 1} \frac{x^2 + 1}{x + 1}$ by replacing numerator and denominator with derivatives?

Solution. No: the initial quotient is $2/2$, not an indeterminate form. Direct substitution gives 1. The derivative quotient would give 2, showing why the hypotheses matter.