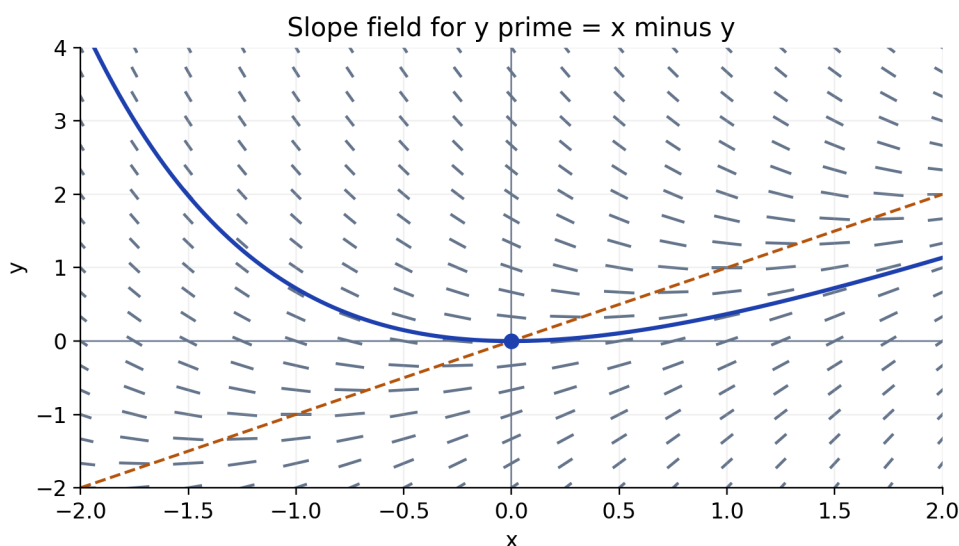


Lesson 6.1 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

1. For $y' = x - y$, identify zero, positive, and negative slopes.

Solution. Slopes are zero on $y = x$, positive below that line where $x - y > 0$, and negative above it. A solution curve must follow these directions.



The dashed diagonal has zero slopes; the blue example solution follows the field through $(0, 0)$.

2. **BC extension.** Use two Euler steps of $h = 1/2$ for $y' = x + y$, $y(0) = 1$, to estimate $y(1)$.

Solution. At $(0, 1)$ the slope is 1, so $y_1 = 3/2$. At $(1/2, 3/2)$ the slope is 2, so $y_2 = 5/2$. Thus $y(1) \approx 5/2$.

3. Verify $y = 2e^{-x}$ as a solution of $y' = -y$, $y(0) = 2$.

Solution. The derivative is $-2e^{-x} = -y$ and the initial value is 2, so both conditions are satisfied.

Lesson 6.2 · Worked solutions

Show the method, check hypotheses, and justify your conclusion.

4. Solve $y' = 3x^2y$ with $y(0) = 2$.

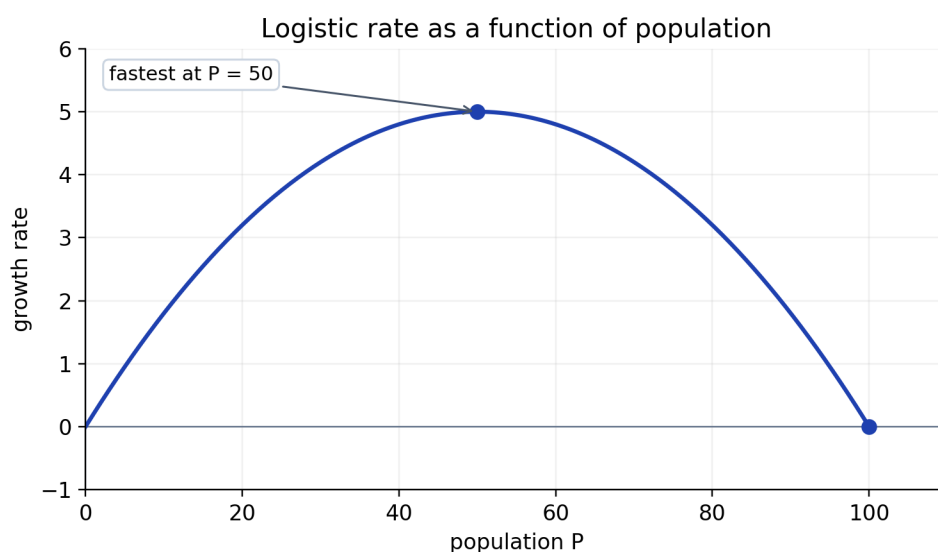
Solution. Integrate $dy/y = 3x^2 dx$ to get $\ln|y| = x^3 + C$. The initial-value solution is $y = 2e^{x^3}$. The equilibrium $y = 0$ exists but does not satisfy this initial value.

5. An amount starts at 80 and has half-life 4 years. Find its amount at 8 years.

Solution. The model is $80e^{-(\ln 2)t/4}$. At 8 years, two half-lives have elapsed, giving 20.

6. **BC extension.** For $P' = 0.2P(1 - P/100)$, identify equilibria, carrying capacity, and the amount at fastest growth.

Solution. Equilibria are $P = 0$ and $P = 100$. Carrying capacity is 100. The rate is maximal at $P = 50$. Below capacity the positive population grows toward 100; above it, the population decreases.



The rate peaks at population 50; the carrying capacity is the equilibrium population 100.

Lesson 6.3 · Worked solutions

BC extension · Optional for AP Calculus AB

7. Find $\int x \cos x \, dx$.

Solution. Choose $u = x$, $dv = \cos x \, dx$. Then the integral is $x \sin x - \int \sin x \, dx = x \sin x + \cos x + C$.

Differentiation cancels the extra sine terms.

8. Find $\int \ln x \, dx$ for $x > 0$.

Solution. Use $u = \ln x$, $dv = dx$. The result is $x \ln x - x + C$. Its derivative is $\ln x + 1 - 1 = \ln x$.

9. Find $\int \frac{1}{x(x+1)} \, dx$ on intervals excluding -1 and 0 .

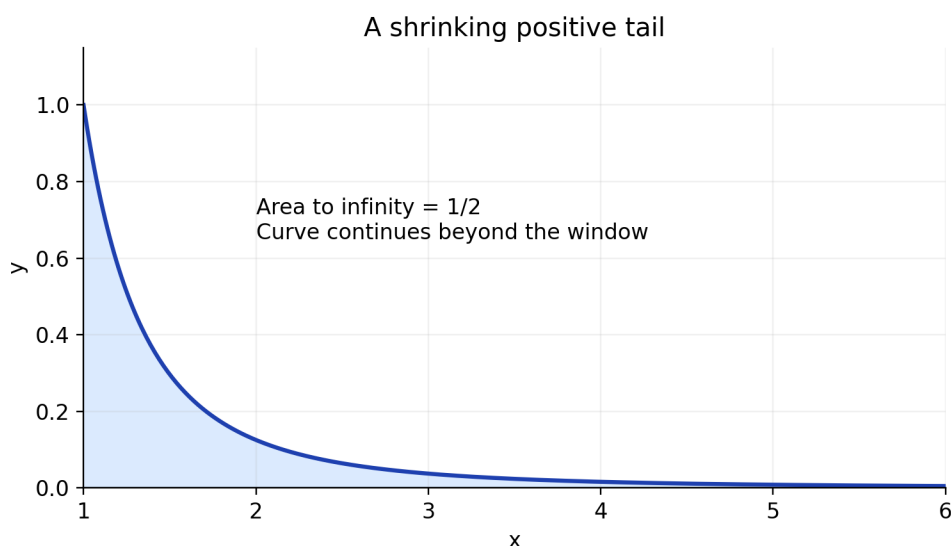
Solution. The decomposition is $1/x - 1/(x+1)$. Integrating gives $\ln|x| - \ln|x+1| + C$.

Lesson 6.4 • Worked solutions

BC extension • Optional for AP Calculus AB

10. Evaluate $\int_1^{\infty} x^{-3} dx$ using a limit.

Solution. The finite integral is $[-1/(2x^2)]_1^b = 1/2 - 1/(2b^2)$. Its limit is $1/2$, so the integral converges.



The total area from 1 to infinity converges to one half; the remaining tail shrinks as the upper bound increases.

11. Determine whether $\int_0^1 x^{-1/2} dx$ converges and find its value if it does.

Solution. Use $\lim_{a \rightarrow 0^+} \int_a^1 x^{-1/2} dx = \lim_{a \rightarrow 0^+} (2 - 2\sqrt{a}) = 2$. It converges.

12. Does $\int_{-1}^1 (1/x) dx$ equal zero?

Solution. No. Split at 0. The left improper integral diverges to negative infinity and the right to positive infinity. Since neither converges, the ordinary improper integral diverges; symmetric cancellation is not its definition.